Q.1. The function $f(x) = 1 + |\sin x|$ is

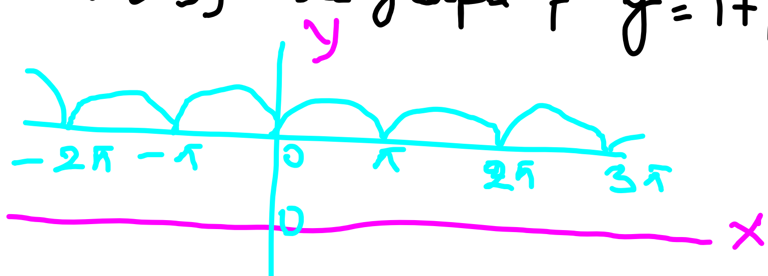
(a) continuous no where

(b) discontinuous at $x = 0$

(c) differentiable no where

(d) non differentiable at infinite number of points

Soln (d) $|\sin x|$ is not differentiable at integral multiple of $\pi$. Hence, $1+|\sin x|$ will also not differentiable at infinite number of those points. It is clear by the graph of $y = 1+|\sin x|$




2. There exists a function $f(x)$ satisfying $f(0) = 1$, $f'(0) = -1$, $f(x) > 0$ for all $x$ and

(a) $f''(x) > 0$ for all $x$

(b) $-1 < f''(x) < 0$ for all $x$

(c) $-2 < f''(x) < -1$ for all $x$

(d) $f''(x) < -2$ for all $x$

Soln. $f(x) > 0 \;\forall\; x \in R$ — (1)

$f(0) = 1$, $f'(0) = -1$ it is possible if $f(x) = e^{-x}$ which satisfy all the three characteristics of function

$\therefore\ f(x) = e^{-x}$

$\therefore\ f'(x) = -e^{-x}$

$f''(x) = e^{-x} > 0\ \forall\ x \in R$

3. The set of all points where the function $f(x) = \dfrac{1}{1+|x|}$ is differentiable is

(a) $(-\infty, \infty)$ (b) $(0, \infty)$ (c) $(-\infty, 0) \cup (0, \infty)$

Soln. $LHD = \lim_{h\to 0} \dfrac{f(0-h) - f(0)}{-h}$

$= \lim_{h\to 0} \dfrac{\frac{1}{1+|0-h|} - 1}{-h} = \lim_{h\to 0} \dfrac{-\cancel{h}}{(1+h)(-\cancel{h})} = 1$

$RHD = \lim_{h\to 0} \dfrac{f(0+h) - f(0)}{h}$

$= \lim_{h\to 0} \dfrac{\frac{1}{1+h} - 1}{h} = \lim_{h\to 0} \dfrac{-\cancel{h}}{\cancel{h}(1+h)} = -1$

$\because$ LHD $\neq$ RHD

$\therefore$ function is not differentiable at $x = 0$

Now, when $x = a > 0$

then

$LHD = \lim_{h\to 0} \dfrac{f(a+h) - f(a)}{h}$

$= \lim_{h\to 0} \dfrac{\frac{1}{1+a+h} - \frac{1}{1+a}}{h}$

$$= \lim_{h\to 0} \frac{-\cancel{h}}{(1+a)(1+a+h)\times \cancel{h}} = \frac{-1}{(1+a)^2}$$

$$RHD = \lim_{h\to 0} \frac{f(a-h)-f(a)}{-h}$$

$$= \lim_{h\to 0} \frac{\frac{1}{1+a-h} - \frac{1}{1+a}}{-h}$$

$$= \lim_{h\to 0} \frac{\cancel{h}}{-\cancel{h}(1+a-h)(1+a)} = \frac{-1}{(1+a)^2}$$

$\because$ LHD = RHD

$\therefore$ $f(x)$ is differentiable for all $x > 0$

Now, when $x = a < 0 \therefore |a| = -a$

LHD

$$\lim_{h\to 0'} \frac{f(a-h)-f(a)}{-h} = \lim_{h\to 0} \frac{\frac{1}{1-(a-h)} - \frac{1}{1-a}}{-h}$$

$$\lim_{h\to 0} \frac{-\cancel{h}}{(1-a)(1-a+h)\times(-\cancel{h})} = \frac{1}{(1-a)^2}$$

$$RHD = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h} = \lim_{h\to 0} \frac{\frac{1}{1-(a+h)} - \frac{1}{1-a}}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{h}}{\cancel{h}(1-a)(1-a-h)} = \frac{1}{(1-a)^2}$$

$\because$ LHD = RHD $\forall\ x \in (-\infty, 0)$

$\therefore$ $f(x)$ is differentiable in

$(-\infty, 0) \cup (0, \infty)$

4. Given that $f(x)$ is differentiable function on the interval $0 \le x \le 5$. Such that $f(0) = 4$ and $f(5) = -1$. If $g'(x) = \frac{f(x)}{x+1}$, then there exists some $c$ where $0 < c < 5$ such that $g''(c)$ is equal to

(a) $-1$ (b) $-\frac{5}{6}$ (c) $-\frac{1}{6}$ (d) $\frac{1}{6}$

Soln: Since $f(x)$ is differentiable function in $0 \le x \le 5$ and

$\therefore g'(x) = \frac{f(x)}{x+1}$ is continuous in $[0,5]$ and differentiable in $(0,5)$

Then applying LMV Theorem

$$g''(c) = \frac{g'(5) - g'(0)}{5-0}$$

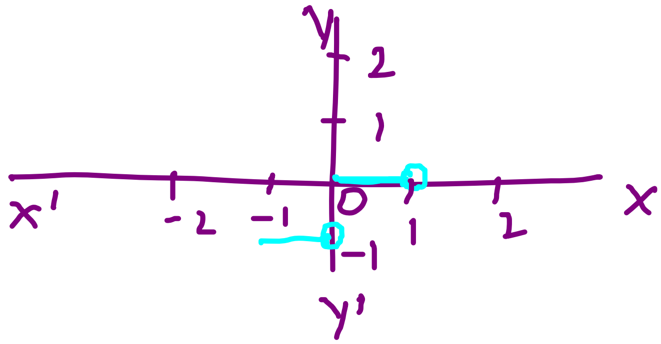
$$= \frac{\frac{f(5)}{6} - \frac{f(0)}{1}}{5} = \frac{\frac{-1}{6} - 4}{5}$$

$$= \frac{-25}{6 \times 5} = -\frac{5}{6} \text{ Ans.}$$

(5) $[x]$ denotes the greatest integer less than or equal to $x$, if $f(x) = [x]$ in $(-1,1)$, then $f(x)$ is

(i) Continuous at $x = 0$
(ii) Continuous in $(-1,0)$
(iii) differentiable in $(-1,1)$
(iv) None of these

Soln: By Graph of $f(x) = [x]$ in $(-1, 1)$




We can easily observe that function is continuous in $(-1, 0)$

6. If $f(x) = \dfrac{\sin(e^{x-2} - 1)}{\log(x-1)}$, $x \neq 2$ is continuous at $x = 2$, Then $f(2) =$

(a) 1 (b) 2 (c) 0 (d) $-2$

Soln: If the function is continuous at $x = 2$

Then $\lim\limits_{x \to 2} f(x) = f(2)$

$$\therefore f(2) = \lim_{x \to 2} \frac{\sin(e^{x-2} - 1)}{\log(x-1)}$$

put $x = 2 + h$

when $x \to 2$, $h \to 0$

$$\therefore f(2) = \lim_{h \to 0} \frac{\sin(e^{h} - 1)}{\log(1+h)}$$

$$= \lim_{h \to 0} \frac{\sin(e^{h} - 1)}{e^{h} - 1} \times \frac{e^{h} - 1}{h} \times \frac{1}{\frac{\log(1+h)}{h}}$$

$$= 1 \times 1 \times 1 = 1 \quad \text{Ans.}$$

(7) $f(x) = x \quad , \quad x < 1$

$= 2 - x, \quad , \quad 1 \le x \le 2$

$= -2 + 3x - x^2 \quad , \quad x \ge 2$ then $f(x)$ is

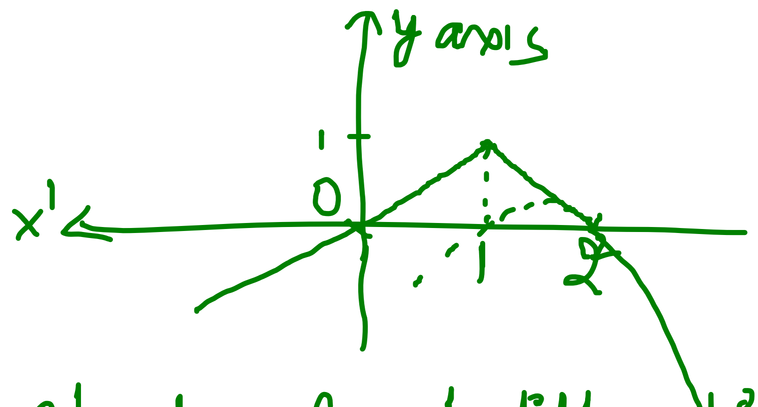
(a) derivable at $x = 1$, $x = 2$

(b) not derivable at $x = 2$

(c) not continuous at $x = 1$, $x = 2$

(d) derivable at $x = 2$ but not at $x = 1$

Soln: from the graph of function $f(x)$




Clearly, $f(x)$ is differentiable at $x = 2$

but not differentiable at $x = 1$, as

at $x = 1$ more than one slope can

be determine.

Q.8 The derivative of the following function

$$y = f(x) = \begin{cases} 2x - 3 & , x \ge 1 \\ 2x + 3 & , x < 1 \end{cases}$$

with respect to $x$, at $x = 1$ is

(a) $(-1)$ (b) $5$ (c) $2$ (d) None of these

Sol<sup></sup>n: Let us check the derivative of function at $x=1$

$$RHD = \lim_{h\to 0} \frac{f(1+h)-f(1)}{h}$$

$$= \lim_{h\to 0} \frac{2(1+h) - 3 - (-1)}{h} = \lim_{h\to 0} \frac{-1+2h+1}{h}$$

$$= \lim_{h\to 0} \frac{2h}{2} = 2$$

$$LHD = \lim_{h\to 0} \frac{f(1-h) - f(1)}{-h}$$

$$= \lim_{h\to 0} \frac{2(1-h)+3 - (-1)}{-h} = \lim_{h\to 0} \frac{6-2h}{-h}$$

$$= -\infty$$

$\because$ LHD $\neq$ RHD

$\therefore f'(x)$ is not defined.

Q.9) If $f(x) = [\sqrt{2}\sin x]$, where [ ] denotes the greatest integer function, then

(a) $f(x)$ is continuous at $x=0$

(b) Maximum value of $f(x)$ is 1 in interval $[-2\pi, 2\pi]$

(c) $f(x)$ is discontinuous at $x = \frac{n\pi}{2} + \frac{\pi}{4}$, $n \in I$

(d) $f(x)$ is discontinuous at $x = n\pi$, $n \in I$

Soln: $\because f(x) = [\sqrt{2}\sin x]$

$\therefore$ $f(x)$ will not be continuous at those points where $\sqrt{2}\sin x$ attain Integral values. This will happen when $\sin x = \pm\frac{1}{\sqrt{2}} \Rightarrow x = \frac{n\pi}{2} + \frac{\pi}{4}, n \in I$

10. Points of discontinuities of function $f(x) = 4x + 7[x] + 2\log(1+x)$, where $[x]$ denotes the integral part of $x$, is

(a) 0 (b) 1 (c) $-\frac{3}{2}$ (d) all of these

Soln: $\because [x]$ is not continuous at Integral value of $x$

$\therefore$ not continuous at $x=0$, $x=1$

$\therefore$ option (a) and (b) true.

Again, $\log(1+x)$ is not defined for $x = -3/2$. Hence, it is not continuous at $x = -3/2$. Hence all of these are correct.

11. If $f(x) = a|\sin^7 x| + b e^{|x|} + c|x|^5$ and if $f(x)$ is differentiable at $x=0$, then which of the following is necessarily true.

(a) $a = b = c = 0$

(b) $a=0,\ b=0,\ c\in R$

(c) $b=c=0,\ a\in R$

(d) $b=0$ & $a$ and $c \in R$

Soln:

$f(x) = a\sin^7 x + b e^{x} + c x^5$

$f(0+h) = a\sin^7 h + b e^{h} + c h^5$

$f(0) = b$

$f(0-h) = a\sin^7 x + b e^{-h} + c h^5$

$\because f(x)$ is differentiable at $x=0$

$$\lim_{h\to 0} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0} \frac{f(0-h)-f(0)}{-h}$$

$$\lim_{h\to 0} \frac{a\sin^7 x + b e^{h} + c h^5 - b}{h}$$

$$= \lim_{h\to 0} \frac{a\sin^7 h + b e^{h} + c h^5 - b}{-h}$$

$$\Rightarrow \lim_{h\to 0} a\frac{\sin h}{h}\sin^6 h + c h^4 + b\frac{(e^h-1)}{h}$$

$$= \lim_{h\to 0} -a\frac{\sin h}{h}\sin^6 h - c h^4 - b\frac{(e^h-1)}{h}$$

$\Rightarrow \quad b = -b \Rightarrow 2b = 0 \Rightarrow b = 0$

$\left[\because \lim_{h\to 0} \frac{\sin h}{h} = 0, \quad \lim_{h\to 0} \frac{e^h - 1}{h} = 1\right]$

Thus $b=0$, Since it is independent of $a$ and $c$ $\therefore$ $a$ and $c$ can be any real number.

12. A even function $f: R \to R$ that is discontinuous at exactly three points is

(a) $g(x) = \begin{cases} 1, & -1 \le x \le 1 \\ 0, & x > 1, x < -1 \end{cases}$

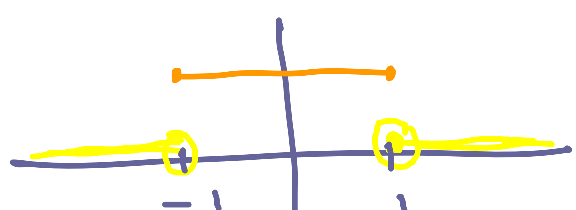
(b) $f(x) = \begin{cases} 0, & \text{if } x < -1, x > 1 \\ 1, & \text{if } -1 \le x < 0, 0 < x \le 1 \\ 2, & \text{if } x = 0 \end{cases}$

(c) $(h\ ) = \begin{cases} 0, & x < 0 \\ 1 & 0 \le x \le 1 \\ 2, & x > 1 \end{cases}$

(d) None of these

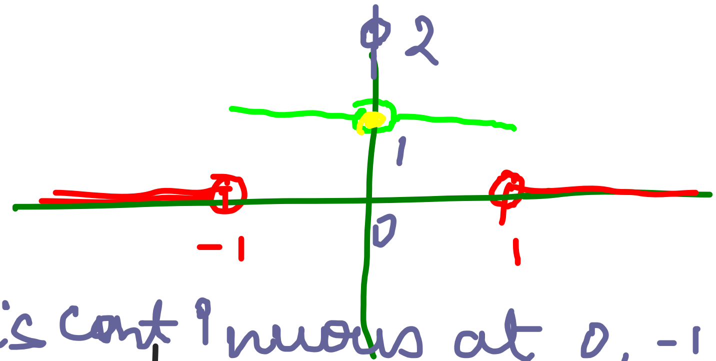
Soln: We can draw graphs of every function

graph of $g(x)$



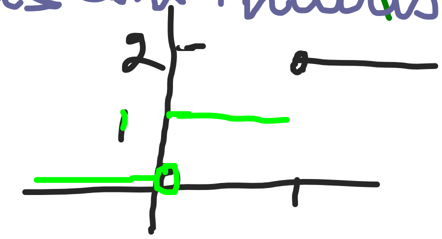

This is discontinuous at two points $-1$ and $1$.

## Graph of $f(x)$




Clearly, $f(x)$ is discontinuous at $0, -1, 1$.

## Graphs of $h(x)$




This graph is discontinuous at 0 and 1.

13. If $f: R \to R$ satisfying $|f(x) - f(y)| \le (x-y)^3$ for all $x, y \in R$ and $f(2) = 5$, then $f(4) =$

(a) 5 (b) 10 (c) 25 (d) None of these

Soln:

$$\frac{|f(x) - f(y)|}{|x-y|^2} \le |x-y|$$

Put $x = y+h$, when $x \to y$, $h \to 0$

$$\lim_{h\to 0} \left| \frac{f(y+h) - f(y)}{h^2} \right| \le \lim_{h\to 0} h$$

$f'(y) = 0 \Rightarrow f(x)$ is constant function

$\therefore f(4) = 5 \quad (\because f(2) = 5)$

14. Let $f(x) = \begin{cases} \frac{x}{2} - 1, & 0 \le x < 1 \\ \frac{1}{2}, & 1 \le x \le 2 \end{cases}$

$$g(x) = (2x+1)(x-k)+3, \quad 0 \le x < \infty$$

Then $g(f(x))$ is continuous at $x = 1$, if $k$ equals

(a) $\frac{1}{2}$ (b) $\frac{11}{6}$ (c) $\frac{1}{6}$ (d) $\frac{13}{6}$

Soln: Here, first of all we should define

$$g(f(x)) = g\left(\frac{x}{2} - 1\right), \quad 0 \le x < 1$$

$$= g\left(\frac{1}{2}\right), \quad 1 \le x \le 2$$

$$\therefore g(f(x)) = 2\left(\left(\frac{x}{2} - 1\right) + 1\right)\left(\frac{x}{2} - k\right) + 3, \quad 0 \le x < 1$$

$$= \left(2 \cdot \frac{1}{2} + 1\right)\left(\frac{1}{2} - k\right) + 3, \quad 1 \le x \le 2$$

$g(f(x)) = h(x)$ say.

$$\therefore h(x) = (x-1)\left(\frac{x}{2} - k\right) + 3, \quad 0 \le x < 1$$

$$= (1 - 2k) + 3, \quad 1 \le x < 2$$

$\because$ $g(f(x))$ i.e $h(x)$ is continuous at $x = 1$

$$\therefore \lim_{h \to 0} f(1-h) = f(1)$$

$$\Rightarrow \lim_{h\to 0} (1-h-1)\left(\frac{1-h}{2}-k\right)+3 = (1-2k)+3$$

$$\Rightarrow \lim_{h\to 0} -h\left(\frac{1-h}{2}-k\right)+3 = 1-2k+3$$

$$\Rightarrow 3 = 4-2k \Rightarrow 2k = 1 \Rightarrow k = \frac{1}{2}$$

15. Which of the following function is differentiable at $x = 0$?

(a) $\cos(|x|) + |x|$ (b) $\cos(|x|) - |x|$

(c) $\sin(|x|) + |x|$ (d) $\sin(|x|) - |x|$

Soln: Check the differentiability by definition of function

for (d)

$$RHD = \lim_{h\to 0} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0} \frac{\sin h - h - 0}{h}$$

$$= \lim_{h\to 0} \frac{\sin h}{h} - 1 = 1-1 = 0$$

$$LHD = \lim_{h\to 0} \frac{f(0-h)-f(0)}{-h} = \lim_{h\to 0} \frac{\sin h - h - 0}{-h}$$

$$= \lim_{h\to 0} -\frac{\sin h}{h} + 1 = -1+1 = 0$$

$\because$ LHD = RHD $\therefore$ $f(x)$ is differentiable at $x=0$

16. A function $f: R \to R$ satisfies the equation $f(x+y) = f(x) \cdot f(y)$ for all $x, y \in R$ and $f(x) \neq 0 \ \forall x \in R$. Let $f'(0) = 2$. Then $f'(x) =$

(a) $f(x)$ (b) $2f(x)$ (c) $-2f(x)$ (d) None of these

Soln: $f(x+y) = f(x) \cdot f(y) \Rightarrow f(x) = e^{2x}$

$\therefore f'(x) = 2e^{2x} \Rightarrow f'(0) = 2 \cdot e^0 = 2$

$\therefore f'(x) = 2f(x)$

17. Let $f(x) = x^3 - x^2 + x + 1$

$g(x) = \max.\{f(t),\ 0 \le t \le x\},\ 0 \le x \le 1$

$= 3 - x, \quad 1 < x \le 2$

Then in $[0, 2]$, the points where $g(x)$ is not differentiable is (are)

(a) 1 (b) 2 (c) 1 and 2 (d) None of these

Soln: Here $f(x) = x^3 - x^2 + x + 1$

$f'(x) = 3x^2 - 2x + 1$

$= 3\left[x^2 - \frac{2}{3}x + \frac{1}{3}\right]$

$= 3\left[\left(x - \frac{1}{3}\right)^2 + \frac{2}{9}\right]$

$= 3\left(x - \frac{1}{3}\right) + \frac{2}{3} > 0$

$\because$ $f(x)$ is increasing function

Now, $f(0) = 1$, $f(2) = 7$, $f(-1) = -2$

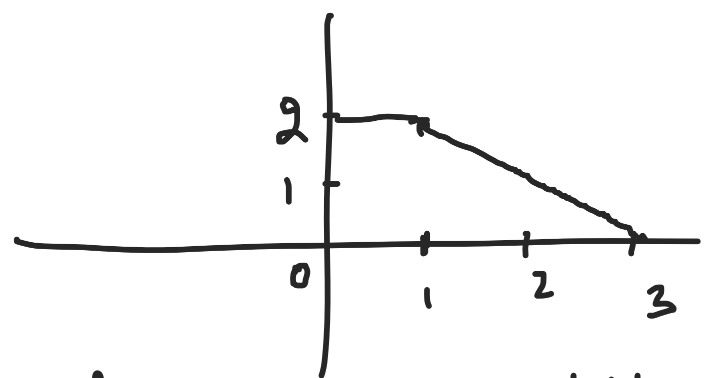
$\because$ $0 \le t \le x \Rightarrow f(0) \le f(t) \le f(x)$

max of $f(t) = f(x) = f(1) = 2$

$\therefore$ $g(x)$ defined as:

$$g(x) = \begin{cases} 2, & 0 \le x \le 1 \\ 3 - x, & 1 < x \le 2 \end{cases}$$

By graph, we can observe it




Clearly, $g(x)$ is not differentiable at $x = 1$

Method 2.

$$RHD = \lim_{h \to 0} \frac{g(1+h) - g(1)}{h} = \lim_{h \to 0} \frac{3-(1+h) - 2}{h}$$

$$= \lim_{h \to 0} \frac{-h}{h} = -1$$

$$LHD \quad \lim_{h \to 0} \frac{g(1-h) - g(1)}{-h} = \frac{2-2}{-h} = 0$$

$\because$ $LHD \ne RHD$

$\therefore$ $f(x)$ is not differentiable at $x = 0$

18. Let $f(x) = [\cos x + \sin x]$, $0 < x < 2\pi$, where $[x]$ denotes the integral part of $x$. Then the number of points of discontinuity of $f(x) =$

(a) 3 (b) 4 (c) 5 (d) 6

Soln: $\because -\sqrt{2} \leq \cos x + \sin x \leq \sqrt{2}$

$$\therefore \quad [\cos x + \sin x] = -2, -1, 0, 1$$

$$\sin x + \cos x = -1 \Rightarrow x = \pi, \frac{3\pi}{2}$$

$$\sin x + \cos x = 0 \Rightarrow x = \frac{3\pi}{4}, \frac{5\pi}{4}$$

$$\sin x + \cos x = 1 \Rightarrow x = 0, \frac{\pi}{2}$$

But $0 < x < 2\pi$, $\therefore$ $x = 0$, can be excluded. and remaining five (5) points are there where $\sin x + \cos x$ attain integral values. Hence $[\sin x + \cos x]$ will not be differentiable at those points.

19. Let
$$f(x) = \begin{cases} x^p \sin \frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

Then $f(x)$ is continuous but not differentiable at $x = 0$, if

(a) $p<0$ (b) $p=0$ (c) $0<p<1$ (d) $p\geq 1$

Soln. A function is differentiable at point $x=0$

if $\lim_{h\to 0} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0} \frac{f(0-h)-f(0)}{-h}$

$\Rightarrow \lim_{h\to 0} \frac{h^p \sin\frac{1}{h} - 0}{h} = \lim_{h\to 0} \frac{(-h)^p \sin\left(-\frac{1}{h}\right) - 0}{-h}$

$\Rightarrow \lim_{h\to 0} h^{p-1} \sin\frac{1}{h} = \lim_{h\to 0} -(-h)^{p-1} \sin\left(\frac{1}{h}\right)$

when $p-1>0$ Then $LHD = RHD = 0$

Then function will be differentiable at that point.

But when $0<p\leq 1$, both the sides gives infinite value. Hence $f(x)$ will not be differentiable at $x=0$

20. Let $f:[0,2]\to[0,2]$ be a continuous function. Then

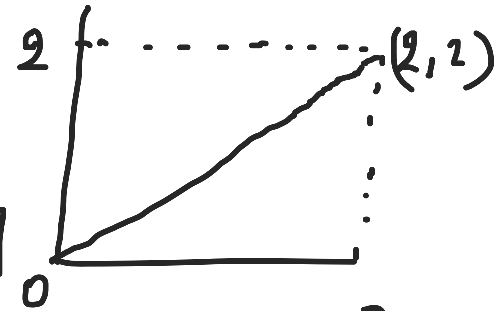
(a) $f(x)$ is differentiable in $[0,2]$

(b) $f(x)=x$ for at least one $x$ in $[0,2]$

(c) $f(x)=-x$ for at least one $x$ in $[0,2]$

(d) None of these

Soln: $\because$ f(x) is continuous function which domain is [0, 2] i.e x varies between 0 to 2. The value of f(x) also lies between 0 to 2. Then the following possibility of graphs of f(x) possible. f(x) = x must satisfy for at least one value of x




All possible graphs of f(x) for which $x \in [0, 2]$ and $f(x) \in [0, 2]$, it must intersect at least once to the line f(x) = x

21. $f(x) = (x-1)^2 \cos \frac{1}{x-1} - |x|, \quad x \neq 0$

$\qquad = -1, \quad x = 0$

Then the points where f(x) is not differentiable is / are

(a) 0 (b) 1 (c) 0, 1 (d) None of these

Soln: Let us check the differentiability at x = 0

$$RHD = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h}$$

$$= \lim_{h \to 0} \frac{(h-1)^2 \cos \frac{1}{h-1} - h - 1}{} = \infty$$

$\therefore f(x)$ is not differentiable at $x=0$

22. The set of all points where the function $f(x) = x|x|$, is differentiable is

(a) $(-\infty,0)\cup(0,\infty)$ (b) $(-\infty,0)$

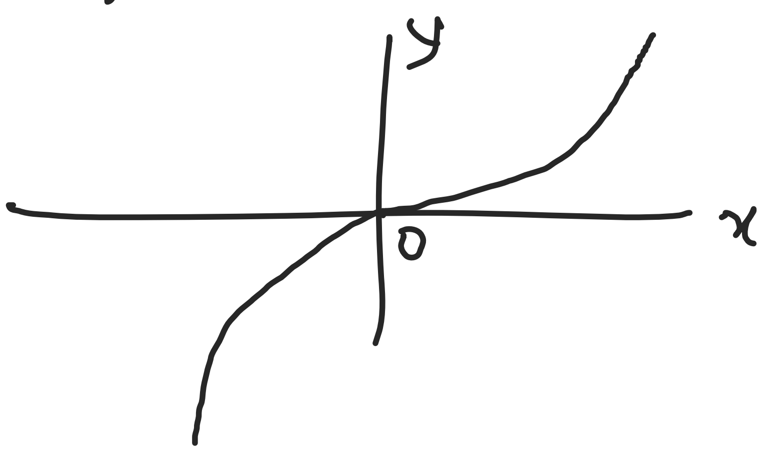
(c) $(-\infty,\infty)$ (d) $(0,\infty)$

Soln. $f(x) = x|x|$

$= x^2, \quad x>0$

$= -x^2, \quad x<0$

Now graph of $f(x)$ will be




Clearly, function is differentiable everywhere. in $(-\infty,\infty)$

23. $f(x) = \begin{cases} x, & \text{if } x \text{ is rational} \\ 1-x, & \text{if } x \text{ is irrational} \end{cases}$, then

(a) $f$ is only right continuous at $x=\frac{1}{2}$

(b) $f$ is only left continuous at $x=\frac{1}{2}$

(c) $f$ is continuous at $x=\frac{1}{2}$

(d) $f$ is discontinuous at all points

Soln. $\lim_{h\to 0} f(\frac{1}{2}+h) = \lim_{h\to 0} 1-(\frac{1}{2}+h) = \lim_{h\to 0} \frac{1}{2}-h = \frac{1}{2}$

$\lim_{h\to 0} f(\frac{1}{2}-h) = \lim_{h\to 0} 1-(\frac{1}{2}-h) = \lim_{h\to 0} \frac{1}{2}+h = \frac{1}{2}$

and $f(\frac{1}{2}) = \frac{1}{2}$ as $\frac{1}{2}$ is rational

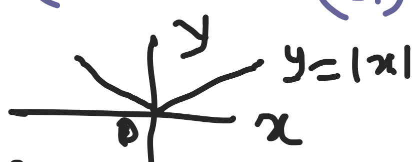
$\therefore \lim_{h\to 0} f(\frac{1}{2}-h) = \lim_{h\to 0} f(\frac{1}{2}+h) = f(\frac{1}{2})$

Hence, $f(x)$ is continuous at $x = \frac{1}{2}$

24. The derivative of $f(x) = |x|$ at $x=0$ is

(a) 1 (b) 2 (c) $-1$ (d) non-existent

Soln. By graph




Clearly, $f(x)$ is not differentiable at $x=0$

25. If $f(x)$ and $|f(x)|$ are both differentiable, Then $f(x)$ is

(a) left continuous at $x=0$

(b) right continuous at $x=0$

(c) not continuous at $x=0$

(d) continuous every where

Soln. $\because$ Every differentiable function is a continuous function. Hence, $f(x)$ is continuous every where.