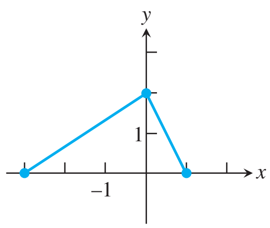
Chapter 1 Additional and Advanced Exercises

Functions and Graphs

1. The graph of f is shown. Draw the graph of each function.

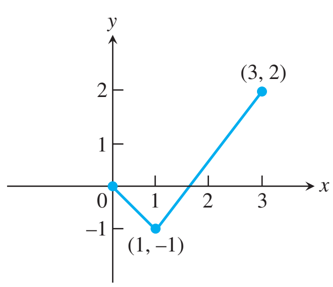
a. $y = f(-x)$ **b.** $y = -f(x)$

c. $y = -2f(x + 1) + 1$ **d.** $y = 3f(x - 2) - 2$



2. A portion of the graph of a function defined on $[-3, 3]$ is shown. Complete the graph assuming that the function is

a. even. **b.** odd.



3. Are there two functions f and g such that $f \circ g = g \circ f$? Give reasons for your answer.

4. Are there two functions f and g with the following property? The graphs of f and g are not straight lines but the graph of $f \circ g$ is a straight line. Give reasons for your answer.

5. If $f(x)$ is odd, can anything be said of $g(x) = f(x) - 2$? What if f is even instead? Give reasons for your answer.

6. If $g(x)$ is an odd function defined for all values of x, can anything be said about $g(0)$? Give reasons for your answer.

7. Graph the equation $|x| + |y| = 1 + x$.

8. Graph the equation $y + |y| = x + |x|$.

Trigonometry

In Exercises 9–14, ABC is an arbitrary triangle with sides a, b, and c opposite angles A, B, and C, respectively.

9. Find b if $a = \sqrt{3}, A = \pi/3, B = \pi/4$.

10. Find $\sin B$ if $a = 4, b = 3, A = \pi/4$.

11. Find $\cos A$ if $a = 2, b = 2, c = 3$.

12. Find c if $a = 2, b = 3, C = \pi/4$.

13. Find $\sin B$ if $a = 2, b = 3, c = 4$.

14. Find $\sin C$ if $a = 2, b = 4, c = 5$.

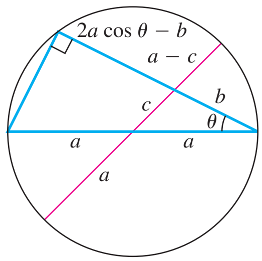
Derivations and Proofs

15. Prove the following identities.

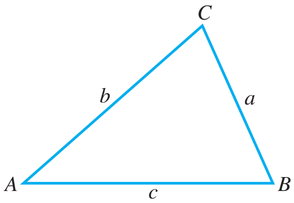
a. $\dfrac{1 - \cos x}{\sin x} = \dfrac{\sin x}{1 + \cos x}$

b. $\dfrac{1 - \cos x}{1 + \cos x} = \tan^2 \dfrac{x}{2}$

16. Explain the following "proof without words" of the law of cosines. (Source: "Proof without Words: The Law of Cosines," Sidney H. Kung, *Mathematics Magazine*, Vol. 63, No. 5, Dec. 1990, p. 342.)



17. Show that the area of triangle ABC is given by $(1/2)ab \sin C = (1/2)bc \sin A = (1/2)ca \sin B$.



18. Show that the area of triangle ABC is given by $\sqrt{s(s - a)(s - b)(s - c)}$ where $s = (a + b + c)/2$ is the semiperimeter of the triangle.

19. Properties of inequalities If a and b are real numbers, we say that ***a* is less than *b*** and write $a < b$ if (and only if) $b - a$ is positive. Use this definition to prove the following properties of inequalities.

If a, b, and c are real numbers, then:

1. $a < b \Rightarrow a + c < b + c$

2. $a < b \Rightarrow a - c < b - c$

3. $a < b$ and $c > 0 \Rightarrow ac < bc$

4. $a < b$ and $c < 0 \Rightarrow bc < ac$
(Special case: $a < b \Rightarrow -b < -a$)

5. $a > 0 \Rightarrow \dfrac{1}{a} > 0$

6. $0 < a < b \Rightarrow \dfrac{1}{b} < \dfrac{1}{a}$

7. $a < b < 0 \Rightarrow \dfrac{1}{b} < \dfrac{1}{a}$

20. Prove that the following inequalities hold for any real numbers a and b.

a. $|a| < |b|$ if and only if $a^2 < b^2$

b. $|a - b| \geq ||a| - |b||$

Generalizing the triangle inequality Prove by mathematical induction that the inequalities in Exercises 21 and 22 hold for any n real numbers $a_1, a_2, \ldots, a_n$. (Mathematical induction is reviewed in Appendix 1.)

21. $|a_1 + a_2 + \cdots + a_n| \leq |a_1| + |a_2| + \cdots + |a_n|$

22. $|a_1 + a_2 + \cdots + a_n| \geq |a_1| - |a_2| - \cdots - |a_n|$

23. Show that if f is both even and odd, then $f(x) = 0$ for every x in the domain of f.

24. a. Even-odd decompositions Let f be a function whose domain is symmetric about the origin, that is, $-x$ belongs to the domain whenever x does. Show that f is the sum of an even function and an odd function:

$$f(x) = E(x) + O(x),$$

where E is an even function and O is an odd function. (*Hint:* Let $E(x) = (f(x) + f(-x))/2$. Show that $E(-x) = E(x)$, so that E is even. Then show that $O(x) = f(x) - E(x)$ is odd.)

b. Uniqueness Show that there is only one way to write f as the sum of an even and an odd function. (*Hint:* One way is given in part (a). If also $f(x) = E_1(x) + O_1(x)$ where E_1 is even and O_1 is odd, show that $E - E_1 = O_1 - O$. Then use Exercise 23 to show that $E = E_1$ and $O = O_1$.)

Grapher Explorations—Effects of Parameters

25. What happens to the graph of $y = ax^2 + bx + c$ as

a. a changes while b and c remain fixed?

b. b changes (a and c fixed, $a \neq 0$)?

c. c changes (a and b fixed, $a \neq 0$)?

26. What happens to the graph of $y = a(x + b)^3 + c$ as

a. a changes while b and c remain fixed?

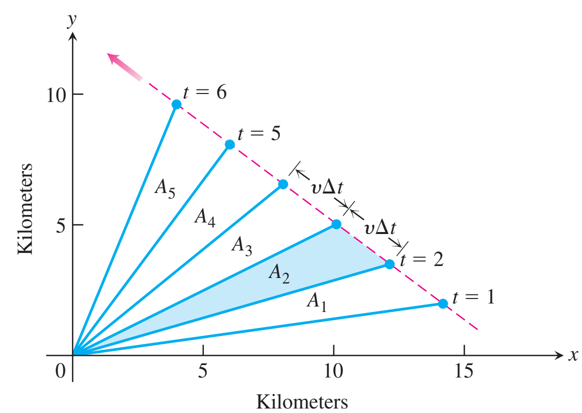
b. b changes (a and c fixed, $a \neq 0$)?

c. c changes (a and b fixed, $a \neq 0$)?

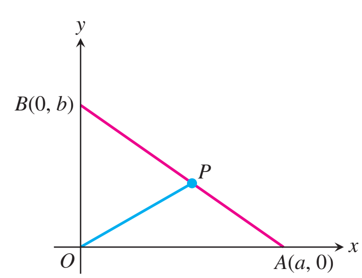
27. Find all values of the slope of the line $y = mx + 2$ for which the x-intercept exceeds $1/2$.

Geometry

28. An object's center of mass moves at a constant velocity v along a straight line past the origin. The accompanying figure shows the coordinate system and the line of motion. The dots show positions that are 1 sec apart. Why are the areas $A_1, A_2, \ldots, A_5$ in the figure all equal? As in Kepler's equal area law (see Section 13.6), the line that joins the object's center of mass to the origin sweeps out equal areas in equal times.



29. a. Find the slope of the line from the origin to the midpoint P, of side AB in the triangle in the accompanying figure $(a, b > 0)$.



b. When is OP perpendicular to AB?